Joe Dinius MATH 527A
HW3 November 29, 2008

1. **Convexity**

A function $f : V \to \mathbb{R}$ from a vector space V to the set of reals is *convex* if it satisfies

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in V,\ t \in [0,1]$.

Also, a set $K \subseteq V$ is *convex* if $x, y \in K \implies tx + (1-t)y \in K$ for all $t \in [0,1]$.

(a) If $f : V \to \mathbb{R}$ is convex, show that the *sub-level sets* defined by

$$E_\alpha = \{x \in V \big| f(x) \leq \alpha\}$$

are convex for all $\alpha \in \mathbb{R}$.

Solution. Let $x, y \in E_\alpha$, $\alpha \in \mathbb{R}$, and $t \in [0,1]$. Therefore $f(x) \leq \alpha$ and $f(y) \leq \alpha$. Since f is convex, the following is true

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\leq t\alpha + (1-t)\alpha = \alpha \\ \implies tx + (1-t)y &\in E_\alpha. \end{aligned}$$

$\therefore$ the sub-level sets E_α are convex for all $\alpha \in \mathbb{R}$.

(b) If V is a normed linear space, show that the unit ball $\{x|\ ||x|| \leq 1\}$.

Solution. The main piece of the proof is to use the triangle inequality of the norm on V. Let $x, y \in$ u.b. where u.b. denotes the unit ball and $t \in [0,1]$. $||x||, ||y|| \leq 1$. Consider $||tx + (1-t)y||$

$$\begin{aligned} ||tx + (1-t)y|| &\leq ||tx|| + ||(1-t)y|| \\ &= t||x|| + (1-t)||y|| \\ &\leq t + (1-t) = t \leq 1 \\ \implies tx + (1-t)y &\in \text{u.b.} \end{aligned}$$

$\therefore$ the unit ball $\{x|\ ||x|| \leq 1\}$ is convex.

2. Two norms $||\cdot||_a$ and $||\cdot||_b$ on a vector space are *equivalent* if they generate equivalent metrics.

(a) Show that the two norms are equivalent if and only if there exist constants c, C such that

$$c||v||_a \leq ||v||_b \leq C||v||_a \quad \forall v \in V.$$

Solution. ($\implies$) Assume the norms are equivalent, then, by the definition of metric spaces

$$\begin{aligned} \forall \epsilon, \epsilon' > 0,\ \exists \delta, \delta' > 0 \text{ s.t.} \quad & ||v - v_0||_a < \delta \implies ||v - v_0||_b < \epsilon \\ \text{and} \quad & ||v - v_0||_b < \delta' \implies ||v - v_0||_a < \epsilon' \end{aligned}$$

we can construct constants

$$\begin{aligned} c &< \frac{\epsilon'}{\delta'} \\ C &> \frac{\epsilon}{\delta} \end{aligned}$$

such that

$$c||v||_a \leq ||v||_b \leq C||v||_a.$$

($\Longleftarrow$) Assume we have constants c, C s.t.

$$c||v||_a \leq ||v||_b \leq C||v||_a.$$

Construct a sequence $x_n \to x$ in l^b. Using the assumption, we get

$$c||x_n - x||_a \leq ||x_n - x||_b \leq C||x_n - x||_a.$$

Pick $\delta' > 0$. Then

$$||x_n - x||_b < \delta' \implies |x_n - x||_a < c\delta' = \epsilon'.$$

Likewise, pick $\delta > 0$ and $x_n \to x$ in l^a, then

$$||x_n - x||_a < \delta \implies |x_n - x||_b < C\delta = \epsilon.$$

$\therefore$ The norms are equivalent since we have

$$\begin{aligned} \forall \epsilon, \epsilon' > 0,\ \exists \delta, \delta' > 0 \text{ s.t.} \quad & ||v - v_0||_a < \delta \implies ||v - v_0||_b < \epsilon \\ \text{and} \quad & ||v - v_0||_b < \delta' \implies ||v - v_0||_a < \epsilon'. \end{aligned}$$

$\therefore$ two norms are equivalent if and only if there exist constants c, C such that

$$c||v||_a \leq ||v||_b \leq C||v||_a \quad \forall v \in V.$$

(b) Show that any two norms on a finite dimensional vector space are equivalent. (The one fact you can use without proof is that the unit l^1 (or any l^p) sphere in $\mathbb{R}^n$ is compact, and in a compact set, every sequence has a convergent subsequence.)

Solution. Let $1 \leq p \leq \infty$. First show that every norm is bounded above by the 1-norm, by the factor of some constant C. To do this, hopefully we get $||\mathbf{x}||_p \leq C||\mathbf{x}||_1$. Construct a basis for $\mathbb{R}^n$, e, s.t. $\mathbf{x} = \sum_{i=1}^n x_i e_i$. Then

$$\left|\left|\sum_{i=1}^n x_i e_i\right|\right|_p \leq \sum_{i=1}^n ||x_i e_i||_p \leq \sum_{i=1}^n |x_i| \; ||\mathbf{e}||_p \leq \max_i |e_i| \sum_{i=1}^n |x_i| = \max_i |e_i| \; ||\mathbf{x}||_1 < \infty.$$

Therefore, every p-norm is bounded by the 1-norm. Now, if $||\mathbf{x}||_1 \leq C||\mathbf{x}||_p$ we are done.

Use proof by contradiction to show that $||\mathbf{x}||_1 \leq C||\mathbf{x}||_p$. Suppose $\exists x_n$, a subsequence of $\mathbf{x}$ for all n s.t. $||x_n||_1 > n||\mathbf{x}||_p$. Define

$$y_n = \frac{x_n}{||\mathbf{x}||_1}$$

which is on the l^1 unit ball: a compact set. Using the hint given; we can use the fact that every compact set contains a convergent subsequence, call it $w_n \to w$ in l^1. Therefore, using earlier result of this exercise,

$$||w_n - w||_p \leq C||w_n - w||_1$$

which shows that $||w_n - w||_p \to 0$. But,

$$\begin{aligned} ||y_n||_p = \frac{||x_n||_p}{||\mathbf{x}||_1} &\leq \frac{1}{n||\mathbf{x}||_1}||x_n||_1 = \frac{1}{n} \\ \text{from above} \quad n||y_n||_p &< ||y_n||_1 \\ \implies ||y_n||_1 &> 1. \end{aligned}$$

However, since $||y_n||_p \to 0$ and $||w_n - w||_p \to 0 \implies w = 0$. However, by our assumption, $||w_n||_1 = 0 \not> 1$. Contradiction.

$\therefore ||\mathbf{x}||_1 \leq C||\mathbf{x}||_p$.

Therefore, two norms are equivalent on a finite dimensional space.

3. **Seminorms** V is a vector space over $\mathbb{R}$. $p : V \to [0, \infty)$ is a seminorm, if it satisfies the triangle inequality and $p(\alpha v) = |\alpha| p(v)$, $\forall v \in V$ and $\alpha \in \mathbb{R}$.

(a) Show that p is a convex function.

Solution. Since p is a seminorm, the triangle inequality holds. We can construct $v, w \in V$ s.t. $w = tx$ and $v = (1-t)y$ for some $x, y \in V$, $t \in [0, 1]$. So, by the triangle inequality,

$$p(tx + (1-t)y) \leq p(tx) + p((1-t)y).$$

Use the second property of seminorms given to get

$$\begin{aligned} p(tx + (1-t)y) &\leq p(tx) + p((1-t)y \\ &= |t|p(x) + |1-t|p(y) \\ &= tp(x) + (1-t)p(y). \end{aligned}$$

which is the definition of a convex function.

$\therefore p$ is convex

(b) If $L : V \to \mathbb{R}$ is a linear map, show that $p(v) = |L(v)|$ is a seminorm.

Solution. First show the triangle inequality. Assume $v, w \in V$, then

$$\begin{aligned} |L(v+w)| &= \left|\sum_i \alpha_i(v_i + w_i)\right| \quad \text{since } L \text{ is linear} \\ &= \left|\sum_i \alpha_i v_i + \alpha_i w_i\right| \\ &\leq \left|\sum_i \alpha_i v_i\right| + \left|\sum_i \alpha_i w_i\right| \\ &= |L(v)| + |L(w)| \end{aligned}$$

which shows the triangle inequality holds.

The second property follows; for $a \in \mathbb{R}$

$$\begin{aligned} |L(av)| &= \left|\sum_i \alpha_i(av_i)\right| \\ &= |a|\left|\sum_i \alpha_i v_i\right| \\ &= |a||L(v)|. \end{aligned}$$

Therefore the second property of seminorms holds.

$\therefore p(v) = |L(v)|$ is a seminorm.

(c) If $q : V \to \mathbb{R}$ is a convex function that satisfies $q(\alpha v) = |\alpha| q(v)$, $\forall v \in V$, show that q is a seminorm.

Solution. Let $x, y \in \mathbb{R}$. We only need to show the triangle inequality holds. Since q is convex satisfying $q(\alpha v) = |\alpha| q(v)$,

$$\begin{aligned} q(tx + (1-t)y) &\leq tq(x) + (1-t)q(y) \\ &\leq q(tx) + q((1-t)y). \end{aligned}$$

Let $v, w \in V$, $v = tx, w = (1-t)y$, we get the triangle inequality $q(v+w) \leq q(v)+q(w)$.

$\therefore q$ is a seminorm.

(d) If p_α is a family of seminorms on V, which are indexed by an arbitrary set $\alpha \in A$, show that p is a seminorm on a linear space V_p, where

$$p(v) = \sup_{\alpha \in A} p_\alpha(v), \qquad V_p = \{v \in V | p(v) < \infty\}.$$

Solution. Start with the triangle inequality. We know that

$$p(v+w) = \sup_{\alpha\in A} p_\alpha(v+w) \le \sup_{\alpha\in A}[p_\alpha(v) + p_\alpha(w)] = \sup_{\alpha\in A} p_\alpha(v) + \sup_{\alpha\in A} p_\alpha(w) = p(v) + p(w)$$

which shows the triangle inequality on (V_p, p).

The second property follows almost immediately; let $a \in \mathbb{R}$,

$$p(av) = \sup_{\alpha\in A} p_\alpha(av) = \sup_{\alpha\in A} |a| p_\alpha(v) = |a| \sup_{\alpha\in A} p_\alpha(v) = |a| p(v).$$

$\therefore$ p is a seminorm on V_p.

(e) Find a countable family $L_k, k \in \mathbb{N}$ of linear maps from which we can obtain the l^p norm on $\mathbb{R}^n$ as

$$||\mathbf{x}||_p = \sup_k |L_k(\mathbf{x})|.$$

Solution. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear function. Then we know that $\dim(null(L)) = (n-1)$. We wish to associate every point on the unit ball with a linear function. To do so we will assign every point with a line tangent to the unit ball at that point. First we choose a coordinate on the unit ball or $\mathbb{R}^n$ by choosing the first $(n-1)$ coordinates to be rational numbers, let them be $(a_1, a_2, a_3, \ldots, a_{n-1})$. Next we have the restriction that this point in $\mathbb{R}^n$ must lie on the unit ball, thus we have

$$a_1^p + a_2^p + \cdots + a_{n-1}^p + a_n^p = 1,$$

or,

$$1 - (a_1^p + a_2^p + \cdots + a_{n-1}^p) = a_n^p.$$

The dimension of the null space of a linear function is $(n-1)$, therefore the $(n-1)$ coordinates that we have chosen will uniquely determine the last coordinate $a_n \in \mathbb{R}$. To find the equation of the line, choose a hyperplane $\mathbf{z} \in \mathbb{R}^n$ that lies tangent to the unit ball. This tangent hyperplane can be expressed by $\mathbf{z} = (a - \mathbf{x})$ where $\mathbf{x}$ is the hyperplane from the end of $\mathbf{z}$ through the origin. Thus the equation for this tangent hyperplane is

$$\nabla f \cdot \mathbf{z} = 0.$$

with $f = a_1^p + a_2^p + \cdots + a_{n-1}^p + a_n^p$. Now using calculus in higher dimensions

$$\begin{aligned}
\nabla f \cdot \mathbf{z} &= (pa_1^{p-1} + \ldots pa_n^{p-1}) \cdot (a - \mathbf{x}) \\
&= a_1^p + \cdots + a_n^p - (a_1^{p-1}x_1 + \cdots + a_n^{p-1}x_n) = 0 \\
\implies 1 &= a_1^{p-1}x_1 + \cdots + a_n^{p-1}x_n.
\end{aligned}$$

This last line defines our linear functional, call it L_1. Keep assigning $(n-1)$ coordinates with these linear functionals and indexing L_i every time you do it to build a countable set of L_i's. Note that this is countable since this is a countable union of $(n-1)$ rational numbers. If we take the p-norm to be the supremum of all of these functions we shall attain the irrational coordinates as well since $\mathbb{N}$ is a dense set in $\mathbb{R}$. Therefore,

$$||\mathbf{x}||_p = \sup_k |L_k(\mathbf{x}|$$

4. **Matrix Norms**

(a) Problem 2.3.35 from Flaschka:

(i) For a matrix $\mathbf{A} = (a_{ij})$ on $\mathbb{R}^n$, show that

$$\begin{aligned} ||\mathbf{A}||_\infty &\leq \max_i \sum_{j=1}^n |a_{ij}|, \\ ||\mathbf{A}||_1 &\leq \max_j \sum_{i=1}^n |a_{ij}|. \end{aligned}$$

Solution. Use the definition of the ∞-norm for matrices

$$||\mathbf{A}||_\infty = \max_{i,j} |a_{ij}| \leq \max_i \sum_{j=1}^n |a_{ij}|.$$

For the next inequality, use 1-norm definition and the fact that $\max_j |a_{ij}| \leq \sum_{j=1}^n |a_{ij}|$

$$||\mathbf{A}||_1 = \sum_{j=1}^n \sum_{i=1}^n |a_{ij}| \leq \max_j \sum_{i=1}^n |a_{ij}|.$$

$\therefore$ the inequalities hold.

(ii) Show that there is always an $\mathbf{x} \in \mathbb{R}^n$ with $||\mathbf{x}||_\infty = 1$, for which

$$||\mathbf{Ax}||_\infty = \max_i \sum_{j=1}^n |a_{ij}|.$$

Hence deduce that $||\mathbf{A}||_\infty = \max_i \sum_{j=1}^n |a_{ij}|$.

Solution. Such a vector $\mathbf{x}$ is easy to construct, take $\mathbf{x}$ to be such that we know the kth row is the max row of $\mathbf{A}$

$$x_j = \begin{cases} -1 & \text{if } a_{kj} < 0 \\ 1 & \text{if } a_{kj} \geq 0 \end{cases}$$

Therefore, $||\mathbf{Ax}||_\infty = \max |\mathbf{Ax}|$. From our definition of $\mathbf{x}$ and that k is the max row, the definition of maximum amounts to the maximum of a row sum of $\mathbf{A}$, that is

$$(\mathbf{Ax})_k = ||\mathbf{Ax}||_\infty = \max_i \sum_{j=1}^n |a_{ij}|.$$

We can define the ∞-norm of a matrix then by

$$||\mathbf{A}||_\infty = \max_{||\mathbf{x}||_\infty=1} ||\mathbf{A}\mathbf{x}||_\infty = \max_i \sum_{j=1}^n |a_{ij}|.$$

(iii) Prove in a similar way that $||\mathbf{A}||_1 = \max_j \sum_{i=1}^n |a_{ij}|$.

Solution. Construct $\mathbf{x}$ as a unit vector in some direction, call it k. That is, the k^{th} entry is 1, all others are 0. Pick k such that the matrix-vector product is maximal, that is where a_{ik} is maximal. From the definition of the 1-norm for vectors, we get

$$||\mathbf{A}||_1 = \max_{||\mathbf{x}||_1=1} ||\mathbf{A}\mathbf{x}||_1 = \max_j \sum_{i=1}^n |a_{ij}|.$$

(b) Problem 2.3.39 from Flaschka:

Let $\mathbf{A}$ be an $n \times n$ matrix. Show that

$$||\mathbf{A}||_2 \leq \sqrt{||\mathbf{A}||_1 ||\mathbf{A}||_\infty}.$$

(Hint: Exercise 1.3.33.)

Solution. From exercise 1.3.33, we get the inequality

$$\begin{aligned} ||\mathbf{A}\mathbf{x}||_q &\leq M^{1/p} N^{1/q} ||\mathbf{x}||_q \\ &= ||\mathbf{A}||_\infty^{\frac{1}{p}} ||\mathbf{A}||_1^{\frac{1}{q}} ||\mathbf{x}||_q. \end{aligned}$$

Let $p = q = 2$. The following is true

$$\begin{aligned} ||\mathbf{A}||_2 &= \max_{||\mathbf{x}||_2=1} ||\mathbf{A}\mathbf{x}||_2 \\ &\leq \max_{||\mathbf{x}||_2=1} ||\mathbf{A}||_\infty^{\frac{1}{2}} ||\mathbf{A}||_1^{\frac{1}{2}} ||\mathbf{x}||_2 \\ &= ||\mathbf{A}||_\infty^{\frac{1}{2}} ||\mathbf{A}||_1^{\frac{1}{2}} \\ &= \sqrt{||\mathbf{A}||_\infty ||\mathbf{A}||_1}. \end{aligned}$$

$\therefore$ the conjecture is true.

5. The space c is the spaces of all sequences $\{x_n\}$ that converge as $n \to \infty$ and the space $c_0 \subset c$ is the space of all sequences which converge to 0. These spaces are equipped with the l^∞ norm.

   Determine the duals for the spaces $l^p(\mathbb{R}, \mathbb{N})$ for $1 \leq p < \infty$ and for the spaces c_0, c.

Solution. Claim: The dual of $l^p(\mathbb{R}, \mathbb{N})$ is $l^q(\mathbb{R}, \mathbb{N})$. I can show this by looking at the set of sequences $\{x | x_n = 0 \ \forall n > N\}$. Given x in this set and ϕ a linear map, consider $\phi(x) = |\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N|$. Now determine if the linear map is bounded.

$$\phi(x) = |\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N|$$

$$\leq \left(\sum_{i=1}^{N} |\alpha_i|^q \right)^{\frac{1}{q}} \|x\|_p$$

$$\implies ||\phi||_{\text{ind}} = \sup_{||x||_p = 1} |\phi(x)| \leq ||\alpha||_q$$

where Hölder's inequality is used. Since α is finite dimensional, clearly ϕ is a bounded map. Now, we need to show equality. To do this construct x such that $||x||_p = 1$ and $||\phi(x)||_{\text{ind}} \geq ||\alpha||_q$. The following works

$$\begin{aligned} x^* &= \left\{ x_n^* = \frac{\text{sgn}(\alpha_n)|\alpha_n|^{q-1}}{||\alpha||_q^{q/p}} \middle| n < N \right\} \\ \implies |\phi(x^*)| &= \frac{||\alpha||_q^q}{||\alpha||_q^{q/p}} = ||\alpha||_q \\ \implies ||\phi||_{\text{ind}} &\geq |\phi(x^*)| = ||\alpha||_q. \end{aligned}$$

which implies equality of the induced norm; that is, ϕ induces norm $\| \cdot \|_q$. The argument can be extended in the same fashion to all N via the appropriate definition of ϕ. Therefore the claim is true: $l^q(\mathbb{R}, \mathbb{N})$ is the dual for the space $l^p(\mathbb{R}, \mathbb{N})$ where $q = \frac{p}{p-1}$.

Next, we need to determine the dual for the space c_0. If $x \in c_0$, $\forall \epsilon \quad \exists M$ s.t. $N > M \implies |x_N| < \epsilon$. Since we have l^∞, $\|x\|_\infty < \infty$. Given some linear map $\phi(x) = \sum_{i=1}^{\infty} \alpha_i x_i$, we have the following

$$|\phi(x)| = \left| \sum_{i=1}^{\infty} \alpha_i x_i \right| \leq \sum_{i=1}^{\infty} |\alpha_i x_i| \leq \|\alpha\|_1 \|x\|_\infty$$

$$\implies ||\phi||_{\text{ind}} = \sup_{||x||_\infty = 1} |\phi(x)| \leq ||\alpha||_1$$

The first inequality is from Hölder's inequality taking limiting case of $p = 1$, $q = \infty$. We are justified in using Hölder's inequality since x is a convergent sequence. Hence ϕ is a bounded linear map. Now for equality follow the same steps as above:

$$\begin{aligned} x^* &= \text{sgn}(\alpha_i) \implies ||x^*||_\infty = 1 \\ |\phi(x^*)| &= ||\alpha||_1 \\ \implies ||\phi||_{\text{ind}} &\geq |\phi(x^*)| = ||\alpha||_1. \end{aligned}$$

which, with the boundedness piece shows equality. Therefore the dual of the space c_0 is $l^1(c_0)$.

The conclusion and argument for the dual of the space c follows exactly from above where instead of considering sequences that converge to 0, we will consider sequences that converge to some $x_0 \in \mathbb{R}$. Consider instead the map

$$\phi(x - x_0) = \left|\sum_{i=1}^{\infty} \alpha_i(x_i - x_0)\right| \leq \sum_{i=1}^{\infty} |\alpha_i(x_i - x_0)| \leq \|\alpha\|_1 \|x - x_0\|_\infty.$$

For the same reasons as above ϕ is a bounded linear map with induced norm $||\cdot||_1$. Therefore the dual of the space c is $l^1(c)$. For the $\phi(x_0)$ part, check the notes on computing duals.

6. Define $p : \mathbb{R}^2 \to [0, \infty)$ by

$$p(x,y) = \begin{cases} |x| + |y| & xy \geq 0 \\ \sqrt{x^2 + y^2} & \text{otherwise} \end{cases}$$

(a) Show that p is a norm on $\mathbb{R}^2$.

Solution. Need to show 3 properties: (i) positive-definiteness, (ii) positive homogeneity and (iii) triangle inequality. Check the notes on computing duals for additional info.

(i) Positive-definiteness follows directly from $|\cdot|$ on $\mathbb{R}$ and $\sqrt{(\cdot)^2 + (\cdot)^2}$ on $\mathbb{R}^2$. So, $p(x, y) = 0 \implies x = y = 0$.

(ii) Positive homogeneity follows since

$$\begin{aligned} p(ax, ay) &= \begin{cases} |ax| + |ay| & xy \geq 0 \\ \sqrt{(ax)^2 + (ay)^2} & \text{otherwise} \end{cases} = \begin{cases} |a|(|x| + |y|) & xy \geq 0 \\ |a|\sqrt{x^2 + y^2} & \text{otherwise} \end{cases} \\ &= |a|p(x, y). \end{aligned}$$

(iii) Triangle inequality

$$\begin{aligned} p(x_1 + x_2, y_1 + y_2) &= \begin{cases} |x_1 + x_2| + |y_1 + y_2| & xy \geq 0 \\ \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} & \text{otherwise} \end{cases} \\ &\leq \begin{cases} (|x_1| + |y_1|) + (|x_2| + y_2|) & xy \geq 0 \\ \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} & \text{otherwise} \end{cases} \\ &= p(x_1, y_1) + p(x_2, y_2) \end{aligned}$$

from properties of $l^1(\mathbb{R}^2, \mathbb{N})$ and $l^2(\mathbb{R}^2, \mathbb{N})$.

$\therefore$ p is a norm.

(b) Determine the dual space (State clearly any identifications you make between spaces of functions in $\mathbb{R}^2$.)

Solution. Consider a 2-d linear map $\phi : \mathbb{R}^2 \to \mathbb{R}$. Steps follow the same as in the previous problem. Let $\mathbf{x} = (x, y)$.

$$|\phi(\mathbf{x})| = \left|\sum_{i=1}^{2} \alpha_i x_i\right| \leq \sum_{i=1}^{2} |\alpha_i x_i| \leq \begin{cases} \|\alpha\|_\infty \|\mathbf{x}\|_1 & xy \geq 0 \\ \|\alpha\|_2 \|\mathbf{x}\|_2 & \text{otherwise} \end{cases}$$

To show equality, follow the same steps as in the previous exercise. Construct any $\mathbf{x}^*$ s.t. the desired norm (1- or 2- in this case) on the unit ball to get the following

$$\begin{aligned} |\phi(\mathbf{x}^*)| &= \begin{cases} ||\alpha||_\infty & xy \geq 0 \\ ||\alpha||_2 & \text{otherwise} \end{cases} \\ \implies ||\phi||_{\text{ind}} &\geq |\phi(\mathbf{x}^*)| = \begin{cases} ||\alpha||_\infty & xy \geq 0 \\ ||\alpha||_2 & \text{otherwise} \end{cases} \end{aligned}$$

Since the object $\alpha \in \mathbb{R}^2$, the dual space is

$$\begin{cases} l^\infty(\mathbb{R}^2) & xy \geq 0 \\ l^2(\mathbb{R}^2) & \text{otherwise} \end{cases}$$